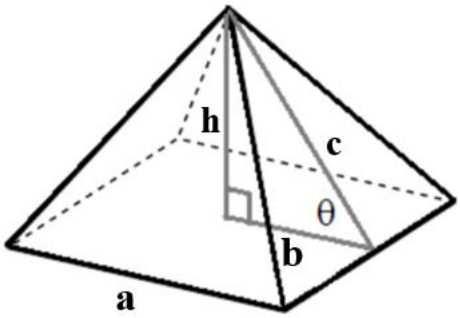
SL Paper 1 Mock A 2020 - WORKED SOLUTIONS v3

Section A

1. The volume of the pyramid is given by



$$V=\frac{1}{3}Ah \Rightarrow 36\sqrt{3}=\frac{1}{3}36h \Rightarrow h=3\sqrt{3}$$

$$b=\frac{a}{2}=\frac{\sqrt{36}}{2}=3$$

$$c=\sqrt{h^2+b^2}=\sqrt{\left(3\sqrt{3}\right)^2+3^2}=\sqrt{27+9}=\sqrt{36}=6$$

$$\Rightarrow \sin\theta=\frac{h}{c} \Rightarrow \theta=\sin^{-1}\left(\frac{h}{c}\right)=\sin^{-1}\left(\frac{3\sqrt{3}}{6}\right)=\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

Hence, $\theta=60°$ $\left[\text{or } \theta=\frac{\pi}{3}\right]$.

2. $\mathrm{P}(A|B)=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)} \Rightarrow \mathrm{P}(B)=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(A|B)}=\frac{1}{5}\cdot\frac{10}{3}=\frac{2}{3}$

$$\mathrm{P}(A)=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B|A)}=\frac{\frac{1}{5}}{\frac{1}{2}}=\frac{2}{5}$$

$$\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$$

$$=\frac{2}{5}+\frac{2}{3}-\frac{1}{5}=\frac{13}{15}$$

Hence, $\mathrm{P}(A\cup B)=\frac{13}{15}$

3. (a) $m=10b+a$

(b) $n-m=10a+b-(10b+a)=9a-9b$

$\Rightarrow \frac{n-m}{9}=a-b\in\mathbb{Z}$ since a and b are integers

Therefore, $n-m$ is divisible by 9.

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4. $h'(x) = x\sqrt{1-x^2} = x\left(1-x^2\right)^{\frac{1}{2}} \Rightarrow h(x) = \int x\left(1-x^2\right)^{\frac{1}{2}}\,dx$

Let $u = 1-x^2 \Rightarrow du = -2x\,dx \Rightarrow -\frac{1}{2}du = x\,dx$

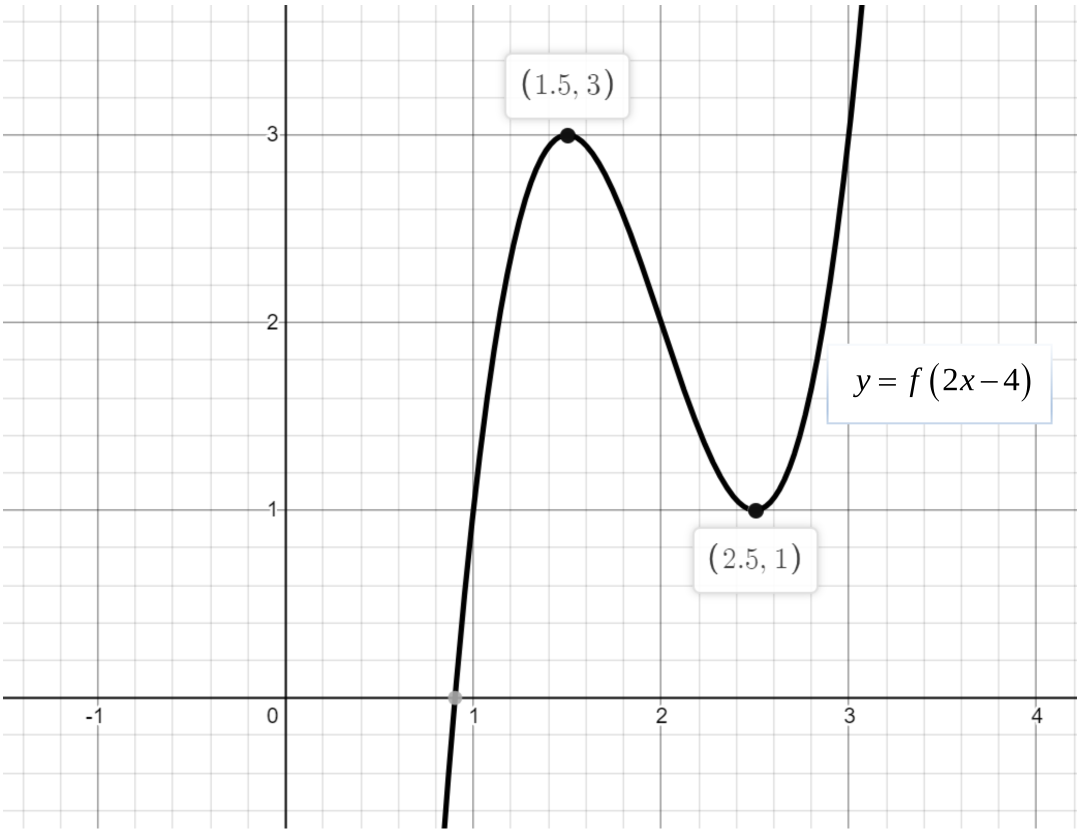
Then,

$$h(x) = -\int \frac{u^{\frac{1}{2}}}{2}\,du = -\frac{u^{\frac{3}{2}}}{3} + C = -\frac{\left(1-x^2\right)^{\frac{3}{2}}}{3} + C$$

$$h(0) = \frac{2}{3}:\; h(0) = -\frac{\left(1-(0)^2\right)^{\frac{3}{2}}}{3} + C = \frac{2}{3} \Rightarrow C = \frac{2}{3} + \frac{1}{3} = 1$$

Hence, $h(x) = -\frac{\left(1-x^2\right)^{\frac{3}{2}}}{3} + 1 \qquad \left[\text{or } -\frac{1}{3}\sqrt{\left(1-x^2\right)^3} + 1\right]$

5. The graph of $y = f\left[2(x-2)\right]$ is formed by a sequence of two transformations of the graph of $f(x)$; a horizontal shrink (with respect to the y axis) by a factor of $\frac{1}{2}$, followed by a horizontal translation 2 units to the right



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6. (a) $\ln x + \ln(x-2) - \ln(x+4) = 0$

$\Rightarrow \ln(x(x-2)) - \ln(x+4) = 0 \Rightarrow \ln(x(x-2)) = \ln(x+4)$

$\Rightarrow x(x-2) = x+4 \Rightarrow x^2 - 3x - 4 = 0$

$\Rightarrow (x-4)(x+1) = 0 \Rightarrow x = 4,\ x = -1$

Verify solutions in original equation:

When $x = 4$: $\ln(4(4-2)) - \ln(4+4) = \ln(8) - \ln(8) = 0$; true.

When $x = -1$, The terms $\ln(x)$ and $\ln(x-2)$ are undefined; false.

Hence, $x = 4$

(b) $\log_3(4x^2 - 5x - 6) = 1 + 2\log_3 x$

$\Rightarrow \log_3(4x^2 - 5x - 6) = 1 + \log_3(x^2) = \log_3(3) + \log_3(x^2)$

$\Rightarrow \log_3(4x^2 - 5x - 6) = \log_3(3x^2) \Rightarrow 4x^2 - 5x - 6 = 3x^2$

$\Rightarrow x^2 - 5x - 6 = 0 \Rightarrow (x-6)(x+1) = 0 \Rightarrow x = 6,\ x = -1$

Verify solutions in original equation:

When $x = 6$:

$\log_3\left(4(6)^2 - 5(6) - 6\right) = \log_3\left(3(6)^2\right) \Rightarrow \log_3(108) = \log_3(108)$ True.

When $x = -1$, The term $\ln(x)$ is undefined; false.

Hence, $x = 6$

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Section B

7. (a) $f'(x) = \cos x - \sin x$

(b) (i) At maxima and minima:

$f'(x) = \cos x - \sin x = 0 \Rightarrow \cos x = \sin x$

$\Rightarrow \frac{\sin x}{\cos x} = \frac{\cos x}{\cos x} \Rightarrow \frac{\sin x}{\cos x} = 1 \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4} + n\pi,\ n \in \mathbb{Z}$

i.e. the graph $f(x) = \cos x + \sin x$ has maxima and minima at $x = \frac{\pi}{4} + n\pi$

When $x = \frac{\pi}{4}$: $f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$

Hence, $\mathrm{A}(p, q) = \mathrm{A}\left(\frac{\pi}{4}, \sqrt{2}\right)$

(ii) $f''(x) = -\sin x - \cos x$

If A is a maximum, then:

$f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2} < 0$

Hence, A is a maximum.

(c) At B, $x = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$

When $x = \frac{5\pi}{4}$: $y = \cos\frac{5\pi}{4} + \sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$

Hence, $\mathrm{B}(x, y) = \mathrm{B}\left(\frac{5\pi}{4}, -\sqrt{2}\right)$

(d) $r = \sqrt{2},\ c = \frac{\pi}{4}$

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8. (a) $\mathrm{P}(\text{1st red ball}) = \frac{4}{6},\ \mathrm{P}(\text{2nd red ball}) = \frac{3}{5} \ \Rightarrow\ \mathrm{P}(\text{2 red balls}) = \frac{4}{6} \cdot \frac{3}{5} = \frac{12}{30} = \frac{2}{5}$

$\mathrm{P}(\text{1st yellow ball}) = \frac{2}{6},\ \mathrm{P}(\text{2nd yellow ball}) = \frac{1}{5} \ \Rightarrow\ \mathrm{P}(\text{2 yellow balls}) = \frac{2}{6} \cdot \frac{1}{5} = \frac{2}{30} = \frac{1}{15}$

$\mathrm{P}(\mathrm{RY}) = \frac{4}{6} \cdot \frac{2}{5} = \frac{4}{15},\ \mathrm{P}(\mathrm{YR}) = \frac{2}{6} \cdot \frac{4}{5} = \frac{4}{15}$

$\Rightarrow\ \mathrm{P}(\text{1 red and 1 yellow}) = \mathrm{P}(\mathrm{RY}) + \mathrm{P}(\mathrm{YR}) = \frac{4}{15} + \frac{4}{15} = \frac{8}{15}$

(b) $\mathrm{P}(\text{2 red} \cap \mathrm{A}) = \mathrm{P}(\text{2 red})\mathrm{P}(\mathrm{A}) = \frac{1}{10} \cdot \frac{2}{6} = \frac{1}{30}$

$\mathrm{P}(\text{2 red} \cap \mathrm{B}) = \mathrm{P}(\text{2 red})\mathrm{P}(\mathrm{B}) = \frac{2}{5} \cdot \frac{4}{6} = \frac{4}{15}$

$\Rightarrow\ \mathrm{P}(\text{2 red}) = \mathrm{P}\big((\text{2 red} \cap \mathrm{A}) \cup (\text{2 red} \cap \mathrm{B})\big) = \frac{1}{30} + \frac{4}{15} = \frac{9}{30} = \frac{3}{10}$

(c) $\mathrm{P}(\mathrm{A} \mid \text{2 red}) = \dfrac{\mathrm{P}(\mathrm{A} \cap \text{2 red})}{\mathrm{P}(\text{2 red})} = \dfrac{\frac{1}{30}}{\frac{9}{30}} = \frac{1}{30} \cdot \frac{30}{9} = \frac{1}{9}$

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9. (a) $g(x)=\frac{x}{e^{x^2}}=xe^{-x^2}$

$g'(x)=u'v+uv'$ Let $u=x,\ v=e^{-x^2}$, then $u'=1,\ v'=-2xe^{-x^2}$

$\Rightarrow g'(x)=e^{-x^2}-2x^2e^{-x^2}$

At maxima and minima:

$g'(x)=e^{-x^2}-2x^2e^{-x^2}=0 \Rightarrow e^{-x^2}\left(1-2x^2\right)=0 \Rightarrow 1-2x^2=0$ since $e^{-x^2}\neq 0$

$\Rightarrow 2x^2=1 \Rightarrow x=\pm\sqrt{\frac{1}{2}}=\pm\frac{\sqrt{2}}{2}$; Since $x\geq 0,\ x=\frac{\sqrt{2}}{2}$

P is a maximum if $g''\left(\frac{\sqrt{2}}{2}\right)<0$ [graph of g concave down]

$g''(x)=-2xe^{-x^2}+4x^3e^{-x^2}-4xe^{-x^2}=\left(4x^3-6x\right)e^{-x^2}$

Since $e^{-x^2}>0$, check if $4x^3-6x<0$

$4\left(\frac{\sqrt{2}}{2}\right)^3-6\frac{\sqrt{2}}{2}=\sqrt{2}-3\sqrt{2}=-2\sqrt{2}<0$

Hence, $P\left(\frac{\sqrt{2}}{2},y\right)$ is the one maximum on g.

(b) $g''(x)=\left(4x^3-6x\right)e^{-x^2}$

Inflexion points exist where $g''(x)=0$ or $g''(x)$ is undefined; $g''(x)$ is defined for all x

Since $e^{-x^2}>0$, if $g''(x)=0$ then $4x^3-6x=0 \Rightarrow 4x^2=6 \Rightarrow x^2=\frac{3}{2} \Rightarrow x=\pm\sqrt{\frac{3}{2}}$

Since $x\geq 0,\ x=\sqrt{\frac{3}{2}}$

When $x<\sqrt{\frac{3}{2}}$, e.g. $x=1$: $g''(1)=\left(4(1)^3-6(1)\right)e^{-(1)^2}=(4-6)e^{-1}<0$

When $x>\sqrt{\frac{3}{2}}$, e.g. $x=2$: $g''(2)=\left(4(2)^3-6(2)\right)e^{-(2)^2}=(32-12)e^{-4}>0$

$g''(x)<0$ for $x<\sqrt{\frac{3}{2}}$, $g''(x)>0$ for $x>\sqrt{\frac{3}{2}}$, hence $g(x)$ has an inflexion point $Q\left(\sqrt{\frac{3}{2}},y\right)$

(c) (i) $x>\sqrt{\frac{3}{2}}$ (ii) $0\leq x<\sqrt{\frac{3}{2}}$

(d) $\int_0^k xe^{-x^2}dx=\frac{1}{2}-\frac{1}{2e^4}$ Let $u=e^{-x^2}$, then $\frac{du}{dx}=-2xe^{-x^2} \Rightarrow du=-2xe^{-x^2}dx$

$\Rightarrow \int_0^k xe^{-x^2}dx=\int_0^k-\frac{du}{2}=\left[-\frac{u}{2}\right]_0^k=\left[-\frac{e^{-x^2}}{2}\right]_0^k=-\frac{e^{-k^2}}{2}+\frac{1}{2}=\frac{1}{2}-\frac{1}{2e^{k^2}}=\frac{1}{2}-\frac{1}{2e^4}$

Hence, $k^2=4 \Rightarrow k=2$